If $p_* \neq 0$ (p_* is formed from p by deleting the i-th component), these become greatest lower and least upper bounds; they are attained when the elements of K satisfy (3.11), where $a_i = a_i^-$ and $a_i = a_i^+$, respectively. If $p_* = 0$, one of the quantities a_i^-, a_i^+, equal to zero, is not attained at a finite K ($a_i \to 0$ as $\|K\| \to \infty$).

We note, moreover, that if $p_* = 0$ our result implies that the absolute values of the diagonal elements of R^{-1} cannot be less than the corresponding values for the matrix $(R+K)^{-1}$.

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STUDY OF THE QUASILINEAR OSCILLATIONS OF MECHANICAL SYSTEMS WITH DISTRIBUTED AND LUMPED PARAMETERS*

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The averaging method is used to study a class of complex oscillatory systems which are described by vector integrodifferential equations with oscillating kernels. These equations arise when analysing mechanical objects which contain elements with distributed and lumped inertial and elastic parameters. Two physically distinct cases of the oscillation of rigid bodies are considered: "resonant" and "non-resonant", as determined by the properties of the mean values of the kernels of the integral terms. In the first case, it is shown that the equations of the first approximation are equivalent to a system of ordinary second-order differential equations, i.e., the order of the system of equations of the motion of a rigid body is doubled. In the second case, sufficient conditions are found for the oscillating initial variables to be slow in the usual sense of the averaging method; the order of the system is then preserved. The conditions are stated, under which the averaging method can be shown to be strictly applicable in asymptotically long time intervals and constructive error estimates are obtained. On the basis of this approach the perturbed horizontal oscillations of a rigid body containing a rectangular cavity with a two-layer heavy fluid which is elastically connected with a fixed base are investigated and qualitative effects are discovered and examined.

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1. Formulation of the problem. When studying certain mechanical oscillatory systems that contain elements with distributed and lumped parameters, we have to solve Cauchy problems for systems of perturbed integrodifferential equations (i.d.e.) of the type

$$z^{\cdot} = Az + \varepsilon \int_{t_0}^{t} W(t, \tau) z(\tau) d\tau + G(t, z, \varepsilon) \tag{1.1}$$

$$z(t_0) = z^{\circ}, \quad \varepsilon \in (0, \varepsilon_0], \quad t - t_0 \in [0, T(\varepsilon)]$$

Here, z is an n-dimensional vector, $z \in D_z$ is an open domain, A is a real constant $(n \times n)$ matrix, whose eigenvalues have zero real parts, and corresponding to multiple eigenvalues we have simple elementary divisors /1, 2/, i.e., the general solution $z_0(t) = Z(t) c$, where $(Z(t)$ is the fundamental matrix and $c = \text{const})$, of the system $z_0^{\cdot} = Az_0$ contains only trigonometric functions of t $(\sin \omega_k t, \cos \omega_k t, 1)$. The function $W(t, \tau)$ is an $(n \times n)$ matrix which is the kernel of the linear integral operator in (1.1), characterizing the influence of the distributed parts of the system on the lumped parts, see Sect.3. The elements of the matrix $W(t, \tau)$ are assumed to be given in the quadrant of $\tau, t \in [t_0, \infty)$, and to be summable uniform almost periodic functions (UAPF) of both arguments /3/. The possibility of independent variation of τ, t is ensured by the time-reversibility of the oscillatory processes in the perturbed (elastic or fluid) media. We assume that the function $G(t, z, \varepsilon)$ is quasiperiodic and summable with respect to t /4, 5/, is continuously differentiable with respect to z, $z \in D_z$, and also, has the form in ε: $G(t, z, \varepsilon) \equiv \varepsilon^{\pi} g(t, z, \varepsilon)$, where $\pi > 0$, and g is uniformly continuous with respect to ε (henceforth the dependence of g on ε is not indicated); thus, $G(t, z, 0) \equiv 0$ for almost all $t \geqslant t_0$, $z \in D_z$.

For applications, it is important to study the behaviour of system (1.1) for small values of the parameter $\varepsilon > 0$ in asymptotically long time intervals: $T(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$. We refine below the order $\pi > 0$ with which G tends to zero and T tends to infinity with respect to ε. The models thus obtained must lead to a significant qualitative change in the behaviour of system (1.1) and be of theoretical and practical interest. One such approach is the averaging method which we use below /5-7/.

To facilitate the use of asymptotic methods, the integrodifferential Cauchy problem (IDCP) (1.1) is reduced to the "standard form" /6/ by the transformation $(z \to x)$, non-singular for $t \in [t_0, \infty)$:

$$x^{\cdot}(t) = \varepsilon \int_{t_0}^{t} K(t, \tau) x(\tau) d\tau + F(t, x(t), \varepsilon), \quad x(t_0) = x^{\circ} \tag{1.2}$$

$$K(t, \tau) = Z^{-1}(t) W(t, \tau) Z(\tau), \quad F(t, x, \varepsilon) = Z^{-1}(t) G(t, Z(t) x, \varepsilon)$$

$$z = Z(t) x, \quad F = \varepsilon^{\pi} f, \quad t - t_0 \in [0, T(\varepsilon)], \quad \varepsilon \in (0, \varepsilon_0]$$

The matrix kernel $K(t, \tau)$ has the properties of the initial kernel $W(t, \tau)$, and the vector function $F(t, x, \varepsilon)$, $x \in D_x$ has the properties of $G(t, z, \varepsilon)$, $z \in D_z$. The evolution of the new "slow" n-vector x is considered in the time interval $t - t_0 \in [0, T(\varepsilon)]$, in which a change occurs that is significant with respect to ε $(x(t) - x^{\circ} \sim 1)$.

Notes. 1°. Under certain conditions /1, 2, 4-7/ the i.d.e. (1.1) reduces to the form (1.2) in which $A = A(t)$ is a periodic matrix, and $G(t, z, \varepsilon) = g_0(t) + \varepsilon^{\pi} g(t, z)$, where $g_0(t)$ is a UAPF.

2°. An IDCP (1.1) or (1.2) is equivalent to a Cauchy problem for a denumerable system of quasilinear ordinary differential equations (o.d.e.) /8, 9/. An averaging method was developed in /8/ as applied to quasilinear oscillatory systems with a denumerable number of degrees of freedom (construction of the so-called one-frequency approximation).

3°. Methods for the approximate study of a linear IDCP of type (1.1), when $A = A(\varepsilon t)$, $W = W(\varepsilon t, \varepsilon \tau)$ are slowly varying matrices, may be found in /10/; and IDCP in the standard form (1.2) with an integral operator whose kernel is non-linear in x were considered under very strict conditions on the means in /11/ etc.

4°. IDCP of type (1.1), (1.2) can be obtained by reduction of systems which describe the oscillations of rigid bodies, connected by boundary conditions with wave motions of systems with distributed parameters (strings, beams, shafts, rods, or a heavy fluid etc.), and also by means of lumped reverse couplings (e.g., elastic elements) with fixed objects. The horizontal periodic motions of a rectangular vessel containing an ideal heavy fluid with a free surface, were considered in /12/.

The averaging method was used in /9/ to solve a linear IDCP of type (1.1), (1.2) when no account is taken of perturbations $(G = F \equiv 0)$, and the problem describes the one-dimensional oscillations of a rigid body that contains a rectangular cavity, entirely filled with a heavy ideal two-layer fluid (a similar example when account is taken of perturbations is considered below in Sect.3).

In the present paper the averaging method is developed for an IDCP of the general type

(1.2). We prove error estimates under different assumptions about the properties of the functions $K(t, \tau)$ and $F(t, x, \varepsilon)$, $\tau, t \in [t_0, \infty)$, $x \in D_x$, $\varepsilon \in (0, \varepsilon_0]$.

2. Construction and proof of averaging-method schemes.

The basic mathematical apparatus for the approximate construction of an IDCP of simpler type than the initial (1.2), and for analysing the solutions and obtaining error estimates, is the Volterra vector integral equation

$$x(t) = x^\circ + \varepsilon \int_{t_0}^{t} L(t, \tau)\, x(\tau)\, d\tau + \int_{t_0}^{t} F(\tau, x(\tau), \varepsilon)\, d\tau \tag{2.1}$$

$$L(t, \tau) \equiv \int_{\tau}^{t} K(\sigma, \tau)\, d\sigma = K_0^{(1)}(\tau)(t - \tau) + L_*(t, \tau),$$

$$K_0^{(1)}(\tau) = \langle K(t, \tau)\rangle_t$$

Eq.(2.1) follows directly from the i.d.e. (1.2) and the properties of the UAPF /3/. In (2.1) the elemnts of the matrices $K_0^{(1)}(\tau)$, $L_*(t, \tau)$ are assumed to be UAPF, continuous and uniformly bounded for $\tau, t \in [t_0, \infty)$. Here and below, the superscript (1) or (2) means that averaging (subscript 0) is performed with respect to the first or second argument; when averaging is performed with respect to both arguments, the superscript is omitted.

We next determine how the "slow" variable x changes with respect to t and ε, depending on the properties of the matrix $L(t, \tau)$ (2.1). We consider two cases: 1) "resonant", when $K_0^{(1)}(\tau) \not\equiv 0$, and 2) "non-resonant", when $K_0^{(1)}(\tau) \equiv 0$, $\tau \in [t_0, \infty)$ and supplementary sufficient conditions hold.

2.1. Averaging in the "resonant" case. Let $K_0^{(1)}(\tau) \not\equiv 0$; then, in the general case, we put $\pi = 1/2$ and we associate with Eq.(2.1) in slow time θ the averaged equation

$$\xi(\Theta) = x^\circ + K_{00} \int_{\theta_0}^{\theta} (\theta - \vartheta)\, \xi(\vartheta)\, d\vartheta + \int_{\theta_0}^{\theta} f_0(\xi(\vartheta))\, d\vartheta \tag{2.2}$$

$$\theta = \sqrt{\varepsilon} t,\ \theta - \theta_0 \in [0, \Theta],\ K_{00} = \langle K_0^{(1)}(\tau)_\tau\rangle,\ f_0(\xi) = \langle f(t, \xi)\rangle_t$$

Here and below, $\Theta = \mathrm{const}$, i.e., $T(\varepsilon) = \Theta/\sqrt{\varepsilon} \sim 1/\sqrt{\varepsilon}$. Differentiating (2.2) with respect to θ, we arrive at the averaged IDCP, corresponding to the initial IDCP (1.2):

$$\xi'(\theta) = K_{00} \int_{\theta_0}^{\theta} \xi(\vartheta)\, d\vartheta + f_0(\xi(\theta)),\quad \xi(\theta_0) = x^\circ \tag{2.3}$$

The IDCP (2.3) can be written as a Cauchy problem for systems of o.d.e.'s of order $2n$ in two ways:

$$\gamma' = \xi,\quad \xi' = K_{00}\gamma + f_0(\xi),\quad \gamma(\theta_0) = 0,\quad \xi(\theta_0) = x^\circ \tag{2.4}$$

$$\gamma'' = K_{00}\gamma + f_0(\gamma'),\quad \gamma(\theta_0) = 0,\quad \gamma'(\theta_0) = x^\circ \tag{2.4'}$$

$$\xi' = \nu,\quad \nu' = K_{00}\xi + f_0'(\xi)\,\nu,\quad \xi(\theta_0) = x^\circ,\quad \nu(\theta_0) = f_0(x^\circ) \tag{2.5}$$

$$\xi'' = K_{00}\xi + f_0'(\xi)\,\xi',\quad \xi(\theta_0) = x^\circ,\quad \xi'(\theta_0) = f(x^\circ) \tag{2.5'}$$

The second form (2.5) or (2.5') of the Cauchy problem is admissible because the function $f_0(\xi)$ is continuously differentiable with respect to $\xi \in D_x$. It follows from (2.4)-(2.5') that, in an interval $\theta - \theta_0 \sim \Theta \sim 1$, the averaged variables ξ, γ, ν receive significant increments of order unity; this is important for applications. The initial variable $x = x(t, \varepsilon)$ varies similarly, since we have:

Assertion 1. Let $K(t, \tau)$, $\tau, t \in [t_0, \infty)$ be a UAPF, and let $f(t, x)$ be quasiperiodic with respect to t and continuously differentiable with respect to x, the derivative being uniformly bounded for $t \in [t_0, \infty)$, $x \in D_x$; also, let the solution $\xi = \xi(\theta)$ of the averaged IDCP (2.3) exist and $\xi(\theta) \in D_x$, $\theta - \theta_0 \in [0, \Theta]$. Then, for sufficiently small $\varepsilon_0 > 0$, $\varepsilon \in (0, \varepsilon_0]$, we have the estimate

$$|x(t, \varepsilon) - \xi(\theta)| \leqslant \sqrt{\varepsilon} C(\Theta),\quad t - t_0 \in [0, \Theta/\sqrt{\varepsilon}] \tag{2.6}$$

where C is a parameter, independent of t, ε, defined constructively by the properties of functions $K(t, \tau)$, $f(t, x)$.

For, putting $x = \xi + \delta$, where δ is the estimated unknown variable (the error), we obtain by means of (2.1)-(2.3) the expression

$$\delta(t)=\varepsilon\int_{t_0}^{t}L(t,\tau)\delta(\tau)\,d\tau+\sqrt{\varepsilon}\int_{t_0}^{t}[f(\tau,\xi+\delta)-f(\tau,\xi)]\,d\tau+ \tag{2.7}$$
$$\sqrt{\varepsilon}\int_{t_0}^{t}[L(t,\tau)-K_{00}(t-\tau)]\,\xi(\vartheta)\,d\vartheta+\sqrt{\varepsilon}\int_{t_0}^{t}[f(\tau,\xi)-f_0(\xi)]\,d\tau$$

The integration is performed in (2.7) in the light of the dependences $\delta=\delta(\tau)$, $\xi=\xi(\vartheta)$, $\vartheta=\sqrt{\varepsilon}\tau$. To estimate $\delta(t)$ and the time interval $t-t_0\sim T(\varepsilon)$, we use Gronwall's lemma /2, 4, 6, 7/; the result is the required estimate (2.6):

$$|\delta(t)|\leqslant\sqrt{\varepsilon}(q\Theta+f_*)\exp[{}^1/_2k_0^{(1)}\Theta^2+\sqrt{\varepsilon}l_*\Theta+f_{0*}'\Theta]\equiv \tag{2.8}$$
$$\sqrt{\varepsilon}C_*(\Theta,\sqrt{\varepsilon})\leqslant\sqrt{\varepsilon}C(\Theta),\quad t-t_0\in[0,\Theta/\sqrt{\varepsilon}],$$
$$\varepsilon\in(0,\varepsilon_0]$$
$$k_0^{(1)}=\sup_{\tau}\|K_0^{(1)}(\tau)\|,\quad l_*=\sup_{\tau,t}\|L_*(t,\tau)\|;$$
$$\tau,t\in[t_0,\infty)$$
$$f_{0*}'=\sup_t\max_{\theta-\theta_0}\left\|\frac{\partial f(t,\xi)}{\partial\xi}\right\|,\quad f_*=\max_{t-t_0}\left|\int_{t_0}^{t}[f(\tau,\xi)-f_0(\xi)]\,d\tau\right|$$
$$q=\max_{\theta-\theta_0}\left|\varepsilon\int_{t_0}^{t}\{[K_0^{(1)}(\tau)-K_{00}](t-\tau)+L_*(t,\tau)\}\,\xi(\vartheta)\,d\tau\right|$$
$$t-t_0\in[0,\Theta/\sqrt{\varepsilon}],\ \theta-\theta_0\in[0,\Theta]$$

The coefficients f_{0*}', f_*, and q in (2.8) depend continuously on the parameters Θ, $\sqrt{\varepsilon}$. The norm of the matrix $\partial f(t,\xi)/\partial\xi$ is bounded, because its elements are continuous and quasi-periodic with respect to t, $t\in[t_0,\infty)$ and are continuous and uniformly bounded with respect to ξ, $\xi\in D_x$. We prove that f_* and q are bounded by writing $f(\tau,\xi)$ and $K_0^{(1)}(\tau)$ as Fourier series /3/ and subsequently integrating by parts and using the estimates $\xi^{\cdot}=O(\sqrt{\varepsilon})$, $t-t_0=O(1/\sqrt{\varepsilon})$. A detailed proof of the estimate for q may be found in /9/, and for f_* in /5/, in the same way as when averaging an o.d.e.

2.2. Averaging in the "non-resonant" case. Now let $K_0^{(1)}(\tau)\equiv 0$; the results of Sect.2.1 with $K_{00}=0$ then remain valid (Assertion 1 and Eqs.(2.2)-(2.5)). The asymptotic analysis of the IDCP (1.2) or integral Eq.(2.1) for a time interval $t-t_0\in[0,\Theta/\varepsilon]$ is of theoretical and practical interest. Thus, let us have

$$K_0^{(1)}(\tau)=\langle K(t,\tau)\rangle_t\equiv 0,\quad F(t,x,\varepsilon)=\varepsilon f(t,x) \tag{2.9}$$

i.e., the degree $\pi=1$ and is matched with the statement of the problem. The kernel $L(t,\tau)$ in (2.1) is then a UAPF for $\tau,t\in[t_0,\infty)$. With the initial Eq.(2.1) we associate the following integral equation with functions L,f, averaged with respect to the argument of integration (analogous with averaging of an o.d.e.):

$$\eta(t)=x^\circ+\varepsilon L_0^{(2)}(t)\int_{t_0}^{t}\eta(\tau)\,d\tau+\varepsilon\int_{t_0}^{t}f_0(\eta(\tau))\,d\tau \tag{2.10}$$
$$L_0^{(2)}(t)=\langle L(t,\tau)\rangle_\tau=\langle L_*(t,\tau)\rangle_\tau,\quad t-t_0\in[0,\Theta/\varepsilon]$$

In the general case, the variables x,η are not slow in the sense of /6, 7/, i.e., we do not have $x^{\cdot},\eta^{\cdot}=O(\varepsilon)$ for $t-t_0=O(1/\varepsilon)$. This can be established by examples.

1°. We consider the special case of Eq.(2.10) when $f_0(\eta)\equiv 0,\eta\in D_x$ and $\langle L_0^{(2)}(t)\rangle_t=L_{00}=0$. The solution of the linear equation is found as

$$\eta(t)=y(t)+\varepsilon L_0^{(2)}(t)\int_0^t y(\tau)\,d\tau\quad(t_0=0) \tag{2.11}$$

where y is the new unknown, obtained by solving the equation

$$y(t)=x^\circ+\varepsilon^2\int_0^t V(t,\tau)\,y(\tau)\,d\tau \tag{2.12}$$
$$V(t,\tau)=\int_\tau^t L_0^{(2)}(t)L_0^{(2)}(\sigma)\,d\sigma$$

The kernel $V(t,\tau)$ in (2.12) is an UAPF with respect to both arguments, and $\|V\|\leqslant v_*=\text{const}$.

Using the results of Sect.2.1 for Eq.(2.12), and putting $y = x^\circ + \delta$, we obtain the estimate

$$|\delta(t)| \leqslant \varepsilon C(\Theta), \quad |y^\cdot| = |\delta^\cdot| \leqslant \varepsilon B(\Theta), \quad t \in [0, \Theta/\varepsilon]$$
$$[|x^\circ| + \varepsilon C(\Theta)][(v_t')_* \Theta + \varepsilon v_*] = B_*(\Theta, \varepsilon) \leqslant B(\Theta), \quad \varepsilon \in (0, \varepsilon_0] \tag{2.13}$$

The constant $(v_t')_*$ upper-bounds the norm of the matrix $\partial V / \partial t$, whose elements must be uniformly bounded. By (2.11), the expressions for $\eta, \eta^\cdot$ have the form for $t \in [0, \Theta/\varepsilon]$:

$$\eta(t, \varepsilon) = x^\circ + \delta(t, \varepsilon) + \theta L_0^{(2)}(t) x + \varepsilon L^{(2)}(t) \int_0^t \delta(\tau, \varepsilon) d\tau \equiv \tag{2.14}$$
$$x^\circ + \theta L_0^{(2)}(t) x^\circ + \varepsilon e(t, \varepsilon), \quad |e| \leqslant e_* = \text{const}$$
$$\eta^\cdot(t, \varepsilon) = y^\cdot(t, \varepsilon) + \theta L_0^{(2)\cdot}(t) x^\circ + \varepsilon L_0^{(2)\cdot}(t) \int_0^t \delta(\tau, \varepsilon) d\tau + \varepsilon L_0^{(2)}(t)[x^\circ + \delta(t, \varepsilon)] \equiv$$
$$\theta L_0^{(2)\cdot}(t) x^\circ + \varepsilon h(t, \varepsilon), \quad |h| \leqslant h_* = \text{const}, \quad t \in [0, \Theta/\varepsilon], \ \theta = \varepsilon t, \ \theta \in [0, \Theta]$$

From (2.14) there follows, for $L_0^{(2)\cdot}(t) \not\equiv 0$, which is equivalent to $L_0^{(2)}(t) \not\equiv L_{00} = 0$, the required estimate: $\eta^\cdot = O(1)$ for $t = O(1/\varepsilon)$.

2°. We will consider another example, when there exist the inverse matrices $(L_0^{(2)}(t))^{-1}$, $t \in [0, \infty)$ and L_{00}^{-1}. By the replacement $\eta \to y$ we obtain the integral equation equivalent to the o.d.e.; for,

$$\eta = [L_{00} - L_0^{(2)}(t)] L_{00}^{-1} x^\circ + L_0^{(2)}(t) L_{00}^{-1} y \tag{2.15}$$

$$y(t) = x^\circ + \varepsilon \int_0^t L_{00}[L_{00} - L_0^{(2)}(\tau)] L_{00}^{-1} x^\circ d\tau + \varepsilon \int_0^t L_{00} L_0^{(2)}(\tau) L_{00}^{-1} y(\tau) d\tau \tag{2.16}$$

By differentiating (2.16) with respect to t we can obtain by the averaging method, to a first approximation in ε, the expressions for y, η:

$$y(t, \varepsilon) = \zeta(\theta) + \delta(t, \varepsilon), \quad |\delta| \leqslant \varepsilon C(\Theta), \quad \theta \in [0, \Theta] \tag{2.17}$$
$$\zeta(\theta) = Z(\theta) x^\circ, \quad Z(\theta) = \exp(L_{00}\theta) \ (Z' = L_{00} Z)$$
$$\eta(t, \varepsilon) = x^\circ + L_0^{(2)}(t) L_{00}^{-1} [Z(\theta) - E] x^\circ + \varepsilon h(t, \varepsilon), \quad |h| \leqslant h_* = \text{const}$$

Thus, y is a slow, and η a fast, variable in the above sense; but the mean of η with respect to t for a fixed θ is a slow variable. It is also been as a result that, with $K_0^{(1)}(\tau) \equiv 0$, the vector solution $x = x(t, \varepsilon)$ of Eq.(2.1) is in general not slow, while at the same time, the vector $\eta = \eta(t, \varepsilon)$ is the solution of Eq.(2.10).

Assertion 2. Under the conditions of Assertion 1 with regard to the properties of smoothness, boundedness, and the existence of a uniform mean with respect to t, τ of the functions $L(t, \tau)$, $f(t, x)$, and the condition $L_0^{(2)}(t) \equiv L_{00} = \text{const}$ for $t \in [t_0, \infty)$, it follows that: 1) the variable $\eta = \eta(t, \varepsilon)$ is slow, in fact: $\eta = \eta(\theta)$, where $\theta = \varepsilon t, \theta - \theta_0 \in [0, \Theta]$, $\Theta \sim 1$; if $\eta(\theta) \in D_x$ here, then 2) with sufficiently small $\varepsilon_0 > 0$ we have

$$|x(t, \varepsilon) - \eta(\theta)| \leqslant \varepsilon C(\Theta), \quad t - t_0 \in [0, \Theta/\varepsilon], \quad \varepsilon \in (0, \varepsilon_0] \tag{2.18}$$

The proof of property 1) follows at once from Eq.(2.10) with $L_0^{(2)}(t) \equiv L_{00}$. We can then introduce the slow argument $\theta = \varepsilon t$, and the following equations, equivalent to the Cauchy problem for a first order o.d.e.:

$$\eta(\theta) = x^\circ + L_{00} \int_{\theta_0}^{\theta} \eta(\vartheta) d\vartheta + \int_{\theta_0}^{\theta} f_0(\eta(\vartheta)) d\vartheta \tag{2.19}$$

$$\eta' = L_{00}\eta + f_0(\eta), \quad \theta - \theta_0 \in [0, \Theta] \tag{2.20}$$

The estimate (2.18), i.e., property 2), is proved by applying Gronwall's lemma for the difference $\delta = x - \eta$; in fact, for the equation

$$\delta(t) = \varepsilon \int_{t_0}^{t} [\Phi(t, \tau, \eta(\vartheta) + \delta(\tau)) - \Phi(t, \tau, \eta(\vartheta))] d\tau + \tag{2.21}$$
$$\varepsilon \int_{t_0}^{t} [\Phi(t, \tau, \eta(\vartheta)) - \Phi_0(t, \eta(\vartheta))] d\tau, \quad \vartheta = \varepsilon\tau$$
$$\Phi(t, \tau, x) = L(t, \tau) x + f(\tau, x), \quad \Phi_0(t, \eta) = \langle \Phi(t, \tau, \eta) \rangle_\tau$$

in the same way as (2.8) for the error δ, we obtain

$$|\delta| \leqslant \varepsilon (r + f_*) \exp[l + (f_\eta')_*] \equiv \varepsilon C_*(\Theta, \varepsilon) \leqslant \varepsilon C(\Theta) \tag{2.22}$$

$$l=\sup_{t,\tau}\|L(t,\tau)\|,\quad r=\sup_t\max_{\theta-\theta_0}\Big|\int_{t_0}^{t}[L(t,\tau)-L_{00}]\,\eta(\vartheta)\,d\tau\Big|$$

$$(f_\eta')_*=\sup_t\max_{\theta-\theta_0}\Big\|\frac{\partial f}{\partial\eta}\Big\|,\quad f_*=\sup_t\max_{\theta-\theta_0}\Big|\int_{t_0}^{t}[f(\tau,\eta(\vartheta))-f_0(\eta(\vartheta))]\,d\tau$$

The same remarks can be made about estimates (2.22) as in Sect.2.1 (see (2.8)).

2.3. Notes. 1°. The condition $L_0^{(2)}(t)\equiv L_{00}$ is satisfied e.g., when a) $L(t,\tau)=L^*(t-\tau)+L^{**}(t,\tau)$, where $\langle L^{**}(t,\tau)\rangle_\tau\equiv 0$, since then $\langle L(t,\tau)\rangle_\tau=\langle L^*(\sigma)\rangle_\sigma=L_{00}$, or b) $L(t,\tau)=L^*(\tau)+L^{**}(t,\tau)$, c) and in other cases, see Sect.3.

2°. Let the disturbance F in Eq.(2.1) be linear with respect to x, i.e., $f(t,x)=M(t)x+\mu(t)$, where M,μ are UAPF's, and let us have the non-resonant case (2.9). Then, by the replacement $x\to y$ of type (2.11), Eq.(2.1) conveniently transforms to

$$x(t)=y(t)+\varepsilon\int_{t_0}^{t}N(t,\tau)\,y(\tau)\,d\tau,\quad N(t,\tau)\equiv L(t,\tau)+M(\tau) \tag{2.23}$$

$$y(t)=x^\circ+\varepsilon^2\int_{t_0}^{t}W(t,\tau)\,y(\tau)\,d\tau+\varepsilon m(t)$$

$$W(t,\tau)\equiv\int_{\tau}^{t}N(t,\sigma)\,N(\sigma,\tau)\,d\sigma,\quad m(t)\equiv\int_{t_0}^{t}\mu(\tau)\,d\tau$$

The following forms are possible for the kernel $W(t,\tau)$: a) $W(t,\tau)$ is a UAPF, $|W|\leqslant w_*=\mathrm{const}$ for $\tau,t\geqslant t_0$; b) $W(t,\tau)$ is written as the sum $W(t,\tau)=W^*(\tau)(t-\tau)+W^{**}$ (this case occurs when $N(t,\tau)\equiv N(\tau)$) or $W(t,\tau)=W^*(t-\tau)(t-\tau)+W^{**}(t,\tau)$, where W^*,W^{**} are UAPF's; c) in the most general case, the matrix function $W(t,\tau)$ can have the form $W(t,\tau)=W^*(t,\tau)(t-\tau)+W^{**}(t,\tau)$, where W^*,W^{**} are UAPF's. In cases a), b), we can show by Assertion 1 and the results of Sect.2.1 that y is a slow variable:

$$y=y(t,\varepsilon)=\zeta(\theta)+\delta(t,\varepsilon),\quad t-t_0\in[0,\Theta/\varepsilon],\quad|\delta|\leqslant\varepsilon C(\Theta) \tag{2.24}$$

$$\zeta(\theta)=x^\circ+W_{00}^*\int_{\theta_0}^{\theta}(\theta-\vartheta)\,\zeta(\vartheta)\,d\vartheta+\mu_0(\theta-\theta_0)$$

The corresponding IDCP (2.3) and the Cauchy problem (2.4), (2.5) for the o.d.e. are obtained in the same way as indicated in Sect.2.1. In the general situation c), the vector y is slow in the sense of /6, 7/ for asymptotic methods (see Sect.2.2). It is not possible to construct its approximate (asymptotic) expression for $t-t_0\sim 1/\varepsilon$, in the same way as in the general case of a perturbing function $f(t,x)$ (1.2), (2.9), which is non-linear with respect to x.

3°. Equivalent to our approach (2.23) is the method of averaging with respect to both arguments of the kernel $N(t,\tau)$ in Eq.(2.1):

$$\eta(\theta)=N_{00}\int_{\theta_0}^{\theta}\eta(\vartheta)\,d\vartheta+\mu_0(\theta-\theta_0)+x^\circ \tag{2.25}$$

The condition for $\eta(\theta)$ to be ε-close to $x(t,\varepsilon)$ for $t-t_0\in[0,\Theta/\varepsilon]$ has the form $N_0^{(2)}(t)=\langle N(t,\tau)\rangle_\tau=N_{00}=\mathrm{const}$, which corresponds to the condition $L_0^{(2)}(t)=L_{00}=\mathrm{const}$ (see Sect.2.2, Assertion 2).

4°. We now take the IDCP (1.2) in which the kernel $K(t,\tau)$ is averaged with respect to the argument of integration τ $(K_0^{(2)}(t)=\langle K(t,\tau)\rangle_\tau)$:

$$y^\cdot(t)=\varepsilon K_0^{(2)}(t)\int_{t_0}^{t}y(\tau)\,d\tau+\varepsilon^\pi f(t,y(t)),\quad y(t_0)=x^\circ \tag{2.26}$$

The variable $y=y(t,\varepsilon)$ is certainly slow in the sense of Sect.2.1: $y^\cdot=O(\sqrt{\varepsilon})$, $t-t_0=O(1/\sqrt{\varepsilon})(\pi={}^1/_2)$. We do not in general have slowness in the sense of Sect.2.2 (with $\pi=1$). Let $y(t,\varepsilon)$ be the solution of IDCP (2.26) with $\pi=1$ for $t-t_0\in[0,\Theta/\varepsilon]$; then, $|x(t,\varepsilon)-y(t,\varepsilon)|\leqslant\varepsilon C(\Theta)$ if

$$\langle S(t,\tau)\rangle_\tau\equiv 0,\quad S(t,\tau)=\int_{\tau}^{t}[K(\sigma,\tau)-K_0^{(2)}(\sigma)]\,d\sigma \tag{2.27}$$

In order that the variable y be slow for $t - t_0 = O(1/\varepsilon)$, we must have $K_0^{(2)}(t) \equiv 0$; this ensures that $x(t, \varepsilon)$ be ε-close to $y(t, \varepsilon)$ if $S(t, \varepsilon) = L(t, \tau)$ has zero mean with respect to $\tau: L_0^{(2)}(t) \equiv 0$, which is a stricter condition than that taken under the conditions of Assertion 2 $(L_0^{(2)}(t) \equiv L_{00})$.

5°. Consider the non-resonant case (2.9). Using Gronwall's lemma in the same way as above, we then obtain the elementary estimate (see /9/)

$$|x(t, \varepsilon) - x^\circ| \leqslant \varepsilon^\beta B(\Theta), \quad t - t_0 \in [0, \Theta/\varepsilon^{1-\beta}], \quad 0 \leqslant \beta \leqslant 1 \tag{2.28}$$

Here, β is a parameter, and B a constant, independent of t, ε. If $\varphi(\theta)$ is the solution of the averaged d.e. $\varphi' = f_0(\varphi)$, $\varphi(\theta_0) = x^\circ$, where $\theta = \varepsilon^\pi t$ is "slow time" and $\theta - \theta_0 \in [0, \Theta]$, $0 < \pi < 1$, we can obtain the estimate

$$|x(t, \varepsilon) - \varphi(\theta)| \leqslant \varepsilon^\gamma B(\Theta), \quad \gamma = \min(\pi, 1-\pi), \quad 0 < \gamma < 1 \tag{2.29}$$

6°. The above results can be extended directly to the IDCP (1.2) with slowly varying parameters (after suitable transformation of the IDCP (1.1))

$$x^\cdot(t) = \varepsilon \int_{t_0}^{t} K(\theta, \vartheta, t, \tau)\, x(\tau)\, d\tau + \varepsilon^\pi f(\theta, t, x(t)) \tag{2.30}$$

$$x(t_0) = x^\circ, \quad \theta = \varepsilon^\pi t, \quad \vartheta = \varepsilon^\pi \tau, \quad 0 < \pi \leqslant 1$$

In accordance with the different assumptions of Sect.2.1 $(\pi = 1/2)$ and Sect.2.2 $(\pi = 1)$ the averaged IDCP's are obtained by scheme (2.4) and (2.20):

$$\xi'(\theta) = \int_{\theta_0}^{\theta} K_{00}(\theta, \vartheta)\, \xi(\vartheta)\, d\vartheta + f_0(\theta, \xi(\theta)), \quad \xi(\theta_0) = x^\circ \tag{2.31}$$

$$\eta'(\theta) = \frac{d}{d\theta} \int_{\theta_0}^{\theta} \Lambda_{00}(\theta, \vartheta)\, \eta(\vartheta)\, d\vartheta + f_0(\theta, \eta(\theta)), \quad \eta(\theta_0) = x^\circ \tag{2.32}$$

$$\Lambda(\theta, \vartheta, t, \tau) = \int_{\tau}^{t} K(\theta, \vartheta, \sigma, \tau)\, d\sigma, \quad \langle \Lambda(\theta, \vartheta, t, \tau) \rangle_\tau = \Lambda_{00}(\theta, \vartheta)$$

7°. For applications it is worth further developing the averaging method for non-linear IDCP's and integral equations, e.g., of the type

$$x^\cdot(t) = F(t, x(t), I, \varepsilon), \quad x(t_0) = x^\circ \tag{2.33}$$

$$I = I[x] = \varepsilon \int_{t_0}^{t} K(t, \tau, x(t), x(\tau))\, d\tau, \quad t - t_0 \in [0, T(\varepsilon)]$$

$$H(t, x(t), I[x], \varepsilon) = 0, \quad \varepsilon \in (0, \varepsilon_0] \tag{2.34}$$

In particular, the vector functions F, H may be linear with respect to the non-linear integral operator I of x. The powers of the small parameter ε in F and H may be different orders π, though they must be matched as indicated in Sects.2.1 and 2.2.

3. Perturbed oscillations of a vessel containing a stratified fluid. We used the above method to study the oscillations of the system described in dimensionless variables by the IDCP (1.1):

$$s^{\cdot\cdot} + s = -\varepsilon \int_0^t \chi(t - \tau)\, s^\cdot(\tau)\, d\tau + \varepsilon^\pi p(t, s, s^\cdot) \tag{3.1}$$

$$s(0) = s^\circ, \quad s^\cdot(0) = 0, \quad t \in [0, \Theta\varepsilon^{-\pi}], \quad \pi = 1/2, 1$$

$$\chi(t) = \sum_{j=0}^{\infty} \frac{\nu_{2j+1}^4}{(2j+1)^4} \cos \nu_{2j+1} t \equiv \sum_{j=0}^{\infty} \chi_j \cos \nu_j^* t; \quad \nu_j \sim \sqrt{j}, \quad j \to \infty$$

The relevant assumptions, the derivation of the IDCP (3.1) with $p \equiv 0$ and its study in the resonant case may be found in /9/. In (3.1), $s = s(t)$ is the vessel displacement, ν_j^* are the eigenfrequencies of the internal waves of the two-layer fluid, and $\varepsilon > 0$ is a small numerical parameter, giving the influence of the fluid oscillations on the system as a whole, whose mass is equal to the sum of the vessel and fluid masses and the apparent additional mass. By the "rotation" transformation $((s, s^\cdot) \to x^T)$, the IDCP (3.1) is reduced to the standard form

$$\begin{Vmatrix} s \\ s^\cdot \end{Vmatrix} = \Pi(t)\, x, \quad x = \begin{Vmatrix} a \\ b \end{Vmatrix}, \quad \Pi(t) = \begin{Vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{Vmatrix} \tag{3.2}$$

$$x^{\cdot}(t)=\varepsilon\int_0^t K(t,\tau)\,x(\tau)\,d\tau+\varepsilon^{\pi}f(t,x(t)),\quad x(0)=x^{\circ}$$

$$a^{\circ}=s^{\circ},\quad b^{\circ}=0,\quad f^T(t,x)=p(t,\Pi(t)\,x)(-\sin t,\cos t)$$

$$K(t,\tau)=\chi(t-\tau)\begin{Vmatrix}-\sin t\sin\tau & \sin t\cos\tau\\ \cos t\sin\tau & -\cos t\cos\tau\end{Vmatrix},\quad \det K(t,\tau)\equiv 0$$

We first consider the strictly resonant case $\nu_k^*=1$ $(k=0,1,2,\ldots)$; the frequency difference $O(\sqrt{\varepsilon})$ can be referred to the perturbation. In accordance with Sect.2.1, we obtain in slow time $\theta=\sqrt{\varepsilon}t$ the averaged IDCP of type (2.3) $(\chi_k=(2k+1)^{-4})$:

$$\xi'(\theta)=K_{00}\int_0^{\theta}\xi(\vartheta)\,d\vartheta+f_0(\xi(\theta)),\quad \xi(0)=x^{\circ}\tag{3.3}$$

$$\xi^T=(\alpha,\beta),\quad K_{00}=-(\chi_k/4)\,\mathrm{diag}\,(1,1),\quad \langle f(t,\xi)\rangle_t=f_0(\xi)$$

In accordance with (2.4), (2.5), the IDCP (3.3) is equivalent to a Cauchy problem for two second-order o.d.e.'s; the integral term reduces to a supplementary "return force". With $f_0\equiv 0$ there are "beats" in the system (see /9/): $s(t,\varepsilon)=s^{\circ}\cos(^1/_2\chi_k\theta)\cos t+O(\sqrt{\varepsilon})$ for $t\in[0,\Theta/\sqrt{\varepsilon}]$. Let $p=ds^3$, i.e., we take account in (3.1) of a typical non-linear added term to the elastic force restoring the vessel; then in (3.3) we have $f_0^T=(^3/_8)\,d\,(\alpha^2+\beta^2)(-\beta,\alpha)$. By introducing the variables ζ,η, where $\zeta'=\alpha,\ \eta'=\beta$, and assuming for simplicity that the initial conditions are zero (see (2.4), (2.4')), we obtain a system of two second-order o.d.e.'s which has the two first integrals $C_{1,2}$:

$$\begin{aligned}&^1/_2(\zeta'^2+\eta'^2)+(\chi_k/8)(\zeta^2+\eta^2)=C_1,\quad C_1={}^1/_2s^{\circ 2}>0\\ &-\zeta'\eta+\eta'\zeta=(^3/_8)\,dC_1(\zeta^2+\eta^2)-(3/128)\,d\chi_k(\zeta^2+\eta^2)^2+\\ &\quad C_2,\quad C_2=0\end{aligned}\tag{3.4}$$

Using relations (3.4), the system is completely integrable; passing to polar coordinates (r,φ), we obtain instead of (3.4):

$$\begin{aligned}&^1/_2(r'^2+r^2\varphi'^2)+(\chi_k/8)\,r^2={}^1/_2s^{\circ 2}\\ &r^2\varphi'=(3/16)\,ds^{\circ 2}r^2-(3/128)\,d\chi_kr^4\\ &r'^2+r^2d^2[(3/16)\,s^{\circ 2}-(3/128)\,\chi_kr^2]^2+(\chi_k/4)\,r^2=s^{\circ 2}\end{aligned}\tag{3.5}$$

The last relation of (3.5) for r, which is integrable in elliptic functions, is obtained by a combination of the first two integrals. After finding $r=r(\theta,s^{\circ})$ from the second equation, we find that $\varphi'(\theta,s^{\circ})=(3/128)\,d\chi_kr^2-(3/16)\,ds^{\circ 2}$. By differentiating r and integrating φ' with respect to θ, we find the unknowns $\alpha=\zeta'(\theta,s^{\circ})$, $\beta=\eta'(\theta,s^{\circ})$, which, in accordance with (3.2) and Sect.2.1, give the coordinate $s(t,\varepsilon)$ and velocity $s^{\cdot}(t,\varepsilon)$ of the vessel with an error of $O(\sqrt{\varepsilon})$ for $t\sim 1/\sqrt{\varepsilon}$.

Now consider the non-resonant case: $\nu_j^*\neq 1+O(\sqrt{\varepsilon})$, $j=0,1,2,\ldots$, and $\pi=1$; then, from (2.20) and Sect.2.2, we have

$$\alpha'=\Lambda\beta-(^3/_8)\,d\beta(\alpha^2+\beta^2),\quad \beta'=-\Lambda\alpha+(^3/_8)\,d\alpha(\alpha^2+\beta^2)\tag{3.6}$$

$$\Lambda=\sum_{j=0}^{\infty}\frac{\chi_j}{1-\nu_j^{*2}}=\sum_{j=0}^{\infty}\frac{\nu_{2j+1}^4}{(2j+1)^4}\frac{1}{1-\nu_{2j+1}^2}\quad(\Lambda\gtrless 0)$$

It follows from (3.6) that, with $p\equiv 0$ $(d=0)$, the influence of internal waves of the two-layer fluid on the oscillations of the system is equivalent to "gyroscopic forces" and a change of frequency by $\varepsilon\Lambda$, which can have either sign $(\Lambda\gtrless 0)$.

If a non-linear (cubic) perturbation is taken into account, we arrive at a similar effect: $s=s^{\circ}\cos[1+\varepsilon\Lambda-(^3/_8)\,\varepsilon ds^{\circ 2}]\,t+O(\varepsilon)$ for $t\in[0,\Theta/\varepsilon]$.

The influence of other perturbations can also be taken into account, e.g., an external almost periodic perturbation, viscous and quadratic friction, or parametric disturbance, etc. Our method is suitable for studying any oscillatory processes of elastic systems that contain elements with lumped and distributed parameters.

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QUADRATIC INTEGRALS OF THE EQUATIONS OF MOTION OF A RIGID BODY IN A LIQUID*

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General integrable cases of the Kirchhoff-Clebsch equations /1, 2/, with a fourth quadratic interval not explicitly dependent on time, are considered. A proof is presented of Steklov's theorem /3/ that the four cases pointed out by Clebsch /2/, Steklov /3/ and Lyapunov /4/ are the only ones for which the equations of inertial motion of a body in a liquid admit of a fourth quadratic integral. An analysis is presented of Lyapunov's statement /4/ that his integrable case may be considered as a limiting case of Steklov's, and Clebsch's third case as a limiting case of his second. It is shown that the fourth integral of the Kirchhoff-Clebsch equations pointed out by Kolosov /6/ does not lead to integrable cases other than those of Steklov and Lyapunov.

In recent years, reports have been published concerning the "discovery" of new integrable cases of the equations of motion of a charged body in a magnetic field, which are isomorphic to the Kirchhoff-Clebsch equations; this runs counter to Steklov's theorem. This prompted the author to undertake an analysis of Steklov's original account /3/, which entirely vindicates the latter's theorem.

1. We consider the problem of the inertial motion of a free body bounded by a simply-connected surface, in a homogeneous, incompressible, ideal liquid, unbounded in all directions, which is in irrotational motion and stationary at infinity.

The kinetic energy of the "body-plus-liquid" system is /2/

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